

# Multi-way spectral partitioning and higher-order Cheeger inequalities

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## Abstract

A basic fact in spectral graph theory is that the number of connected components in an undirected graph is equal to the multiplicity of the eigenvalue zero in the Laplacian matrix of the graph. In particular, the graph is disconnected if and only if there are at least two eigenvalues equal to zero. Cheeger’s inequality and its variants provide an approximate version of the latter fact; they state that a graph has a sparse cut if and only if there are at least two eigenvalues that are close to zero.

It has been conjectured that an analogous characterization holds for higher multiplicities, i.e., there are  $k$  eigenvalues close to zero if and only if the vertex set can be partitioned into  $k$  subsets, each defining a sparse cut. We resolve this conjecture positively. Our result provides a theoretical justification for clustering algorithms that use the bottom  $k$  eigenvectors to embed the vertices into  $\mathbb{R}^k$ , and then apply geometric considerations to the embedding.

We also show that these techniques yield a nearly optimal quantitative connection between the expansion of sets of size  $\approx n/k$  and  $\lambda_k$ , the  $k$ th smallest eigenvalue of the normalized Laplacian, where  $n$  is the number of vertices. In particular, we show that in every graph there are at least  $k/2$  disjoint sets (one of which will have size at most  $2n/k$ ), each having expansion at most  $O(\sqrt{\lambda_k \log k})$ . Louis, Raghavendra, Tetali, and Vempala have independently proved a slightly weaker version of this last result. The  $\sqrt{\log k}$  bound is tight, up to constant factors, for the “noisy hypercube” graphs.

## 1 Introduction

Let  $G = (V, E)$  be an undirected,  $d$ -regular graph. Its normalized Laplacian matrix  $L \in \mathbb{R}^{V \times V}$  is given by  $L = I - \frac{1}{d}A$ , where  $A$  is the adjacency matrix of  $G$ . For the moment, we confine ourselves to unweighted, regular graphs, while the results in the paper are presented for arbitrary weighted graphs, with suitable changes to  $L$ . It is easy to see that  $L$  is a positive semi-definite matrix, and its eigenvalues satisfy  $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{|V|}$ . Elementary arguments show that the number of connected components of  $G$  is precisely the multiplicity of the eigenvalue zero, that is,  $\lambda_k = 0$  if and only if the graph has at least  $k$  connected components.

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Cheeger’s inequality for graphs [AM85, Alo86, SJ89] yields a robust version of this fact for  $k = 2$ . To state it, we introduce some notation. For any subset  $S \subseteq V$ , define the *expansion* of  $S$  to be the quantity

$$\phi_G(S) = \frac{|E(S, \overline{S})|}{d|S|},$$

where  $E(S, \overline{S})$  denotes the set of edges of  $G$  crossing from  $S$  to its complement. We may also define, for every  $k \in \mathbb{N}$ , the *k-way expansion constant*,

$$\rho_G(k) = \min_{S_1, S_2, \dots, S_k} \max\{\phi_G(S_i) : i = 1, 2, \dots, k\},$$

where the minimum is over all collections of  $k$  non-empty, disjoint subsets  $S_1, S_2, \dots, S_k \subseteq V$ . Observe that  $\rho_G(k) = 0$  if and only if  $\lambda_k = 0$ . Cheeger’s inequality offers the following quantitative connection between  $\rho_G(2)$  and  $\lambda_2$ ,

$$\frac{\lambda_2}{2} \leq \rho_G(2) \leq \sqrt{2\lambda_2}. \quad (1)$$

We remark that the left-hand side follows easily, and the non-trivial content of the connection is contained in the right-hand side inequality.

The discrete version of Cheeger’s inequality is proved via a simple spectral partitioning algorithm. Besides being an important theoretical tool, since their inception spectral methods have been used for solving a wide range of optimization problems, from graph coloring [AG83, AK97] to image segmentation [SM00, TM06] to web search [Kle99, BP98].

**Higher-order Cheeger inequalities.** In general, we study higher-order analogs of (1), and develop new multi-way spectral partitioning algorithms. A special case of one of our main theorems (see Section 3.4 and Theorem 4.9) follows. It offers a strong quantitative version of the fact that  $\rho_G(k) = 0 \iff \lambda_k = 0$ .

**Theorem 1.1.** *For every graph  $G$ , and every  $k \in \mathbb{N}$ , we have*

$$\frac{\lambda_k}{2} \leq \rho_G(k) \leq O(k^2)\sqrt{\lambda_k}. \quad (2)$$

This resolves a conjecture of Miclo [Mic08]; see also [DJM12], where some special cases are considered. Independent of our work, Tanaka [Tan12] proved a weaker version of the theorem by showing that

$$\rho_G(k) \leq O(3^k)\sqrt{\lambda_k}.$$

His proof is constructive but, unlike our approach, it does not provide a polynomial-time algorithm for constructing the  $k$  sets. We remark that from Theorem 1.1, it is easy to find a *partition* of the vertex set into  $k$  non-empty pieces such that every piece in the partition has expansion  $O(k^3)\sqrt{\lambda_k}$  (see Theorem 3.8). It is known that a dependence on  $k$  in the right-hand side of (2) is necessary; see Section 4.3.

Moreover, our proof is algorithmic and leads to new algorithms for  $k$ -way spectral partitioning. This provides a theoretical justification for clustering algorithms that use the bottom  $k$  eigenvectors of the Laplacian<sup>1</sup> to embed the vertices into  $\mathbb{R}^k$ , and then apply geometric considerations to the embedding. See [VM03] for a survey of such approaches. As a particular example, consider the work

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<sup>1</sup>Equivalently, algorithms that use the top  $k$  eigenvectors of the adjacency matrix.

of Jordan, Ng and Weiss [NJW02] which applies a  $k$ -means clustering algorithm to the embedding in order to achieve a  $k$ -way partitioning. Our proof of Theorem 1.1 employs a similar algorithm, where the  $k$ -means step is replaced by a random geometric partitioning. It remains an interesting open problem whether  $k$ -means itself can be analyzed in this setting.

**Finding many sets and small-set expansion.** If one is interested in finding slightly fewer sets, our approach performs significantly better.

**Theorem 1.2.** *For every graph  $G$ , and every  $k \in \mathbb{N}$ , we have*

$$\rho_G(k) \leq O(\sqrt{\lambda_{2k} \log k}). \quad (3)$$

*If  $G$  is planar then, the bound improves to,*

$$\rho_G(k) \leq O(\sqrt{\lambda_{2k}}). \quad (4)$$

*More generally, if  $G$  excludes  $K_h$  as a minor, then*

$$\rho_G(k) \leq O(h^2 \sqrt{\lambda_{2k}}).$$

We remark that the bound (3) holds with  $2k$  replaced by  $(1 + \delta)k$  for any  $\delta > 0$ , but where the leading constant now becomes  $\delta^{-3}$ ; see Corollary 4.2. Louis, Raghavendra, Tetali and Vempala [LRTV12] have independently proved a somewhat weaker version of the bound (3), using rather different techniques. Specifically, they show that there exists an absolute constant  $C > 1$  such that  $\rho_G(k) \leq O(\sqrt{\lambda_{Ck} \log k})$ .

In particular, Theorem 1.2 has applications to the small-set expansion problem in graphs, which is fundamentally connected to the Unique Games Conjecture and many other problems in approximation algorithms (see [RS10, RST10]). To capture the expansion of small sets in graphs, we define the value,

$$\varphi_G(k) = \min_{S \leq |V|/k} \phi_G(S).$$

Clearly  $\varphi_G(k) \leq \rho_G(k)$  for every  $k \in \mathbb{N}$ .

Arora, Barak and Steurer [ABS10] prove the bound,

$$\varphi_G(k^{1/100}) \leq O(\sqrt{\lambda_k \log_k n}),$$

where  $n = |V|$ . Note that for  $k = n^\epsilon$  and  $\epsilon \in (0, 1)$ , one achieves an upper bound of  $O(\sqrt{\lambda_k})$ , and this small loss in the expansion constant is crucial for applications to approximating small-set expansion. This was recently improved further [GT12, OW12] by showing that for every  $\alpha > 0$ ,

$$\varphi_G(k^{1-\alpha}) \leq O(\sqrt{(\lambda_k/\alpha) \log_k n}).$$

These bounds work fairly well for large values of  $k$ , but give less satisfactory results when  $k$  is smaller.

Louis, Raghavendra, Tetali and Vempala [LRTV11] proved that

$$\varphi_G(\sqrt{k}) \leq O(\sqrt{\lambda_k \log k}),$$

and conjectured that  $\sqrt{k}$  could be replaced by  $k$ . Theorem 1.2 immediately yields,

$$\varphi_G(k/2) \leq O(\sqrt{\lambda_k \log k}) \quad (5)$$

resolving their conjecture up to a factor of 2 (and actually, as discussed earlier, up to a factor of  $1 + \delta$  for every  $\delta > 0$ ).

Moreover, (5) is quantitatively optimal for the noisy hypercube graphs (see Section 4.3), yielding an optimal connection between the  $k$ th Laplacian eigenvalue and expansion of sets of size  $\approx n/k$ .

It is interesting to note that in [KLPT11], it is shown that for  $n$ -vertex, bounded-degree planar graphs, one has  $\lambda_k = O(k/n)$ . Thus the spectral algorithm guaranteeing (4) partitions such a planar graph into  $k$  disjoint pieces, each of expansion  $O(\sqrt{k/n})$ . This is tight, up to a constant factor, as one can easily see for an  $\sqrt{n} \times \sqrt{n}$  planar grid, in which case the set of size  $\approx n/k$  with minimal expansion is a  $\sqrt{n/k} \times \sqrt{n/k}$  subgrid.

## 1.1 High-dimensional spectral partitioning

We now present an overview of the proofs of our main theorems, as well as explain our general approach to multi-way spectral partitioning. Let  $G = (V, E)$  be an undirected,  $d$ -regular graph. To begin, for any  $f : V \rightarrow \ell_2$ , we recall the *Rayleigh quotient*,

$$\mathcal{R}_G(f) = \frac{\sum_{\{u,v\} \in E} \|f(u) - f(v)\|^2}{d \sum_{u \in V} \|f(u)\|^2}.$$

The Dirichlet version of Cheeger's inequality (see Lemma 2.1) proves that for any  $f : V \rightarrow \ell_2$ , it is possible to find a subset  $S \subseteq \{v \in V : f(v) \neq 0\}$  such that  $\phi_G(S) \leq \sqrt{2\mathcal{R}_G(f)}$ . Thus in order to find  $k$  disjoint, non-expanding subsets  $S_1, S_2, \dots, S_k \subseteq V$ , it suffices to find  $k$  disjointly supported functions  $\psi_1, \psi_2, \dots, \psi_k : V \rightarrow \ell_2$  such that  $\mathcal{R}_G(\psi_i)$  is small for each  $i = 1, 2, \dots, k$ .

In fact, in the same paper that Miclo conjectured the validity of Theorem 1.1, he conjectured that finding such a family  $\{\psi_i\}$  should be possible [Mic08, DJM12]. We resolve this conjecture and prove the following theorem in Section 3.4.

**Theorem 1.3.** *For any graph  $G = (V, E)$  and any  $k \in \mathbb{N}$ , there exist disjointly supported functions  $\psi_1, \psi_2, \dots, \psi_k : V \rightarrow \mathbb{R}$  such that for each  $i = 1, 2, \dots, k$ , we have*

$$\mathcal{R}_G(\psi_i) \leq O(k^6) \lambda_k.$$

To prove this, we start with an orthonormal system of eigenfunctions of the Laplacian,

$$f_1, f_2, \dots, f_k : V \rightarrow \mathbb{R},$$

where  $f_i$  has eigenvalue  $\lambda_i$ . We then construct the embedding  $F : V \rightarrow \mathbb{R}^k$  given by

$$F(v) = (f_1(v), f_2(v), \dots, f_k(v)). \quad (6)$$

Observe that  $\mathcal{R}_G(F) \leq \lambda_k$ .

Thus our goal is now to “localize”  $F$  on  $k$  disjoint regions to produce disjointly supported functions  $\psi_1, \psi_2, \dots, \psi_k : V \rightarrow \mathbb{R}^k$ , each with small Rayleigh quotient. (It is elementary to see that for any map  $\psi : V \rightarrow \mathbb{R}^k$ , there exists some coordinate  $j \in \{1, 2, \dots, k\}$  such that the  $\mathbb{R}$ -valued map  $\tilde{\psi}(v) = \psi(v)_j$  has  $\mathcal{R}_G(\tilde{\psi}) \leq \mathcal{R}_G(\psi)$ .) In order to ensure that  $\mathcal{R}_G(\psi_i)$  is small for each  $i$ , we

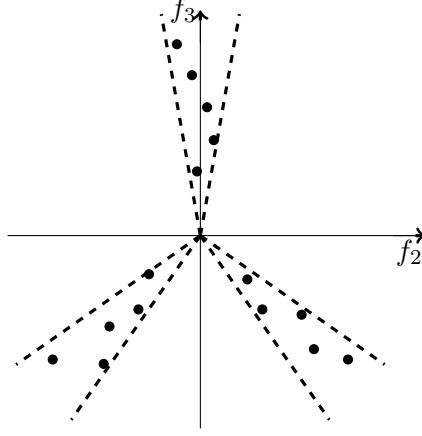


Figure 1: Partitioning according to the radial distance.

must ensure that each region captures a large fraction of the  $\ell^2$  mass of  $F$ , and that our localization process is sufficiently smooth.

**Isotropy and spreading.** The first problem we face is that, in order to find  $k$  disjoint regions each with large  $\ell^2$  mass, it should be that the  $\ell^2$  mass of  $F$  is sufficiently well-spread. This follows from the following *isotropy* property of  $F$  (see Lemma 3.2): For any vector  $x \in S^{k-1}$  (the unit sphere of  $\mathbb{R}^k$ ),

$$\sum_{v \in V} \langle x, F(v) \rangle^2 = 1. \quad (7)$$

On the other hand, it is straightforward to check that,

$$\sum_{v \in V} \|F(v)\|^2 = k,$$

thus it is impossible for the  $\ell^2$  mass of  $F$  to “concentrate” along fewer than  $k$  directions  $x_1, x_2, \dots, x_k \in S^{k-1}$ .

A natural approach would be to find (at least)  $k$  such directions, and then define,

$$\psi_i(v) = \begin{cases} F(v) & \text{if } F(v) \text{ has large projection on } x_i \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately, this sharp cutoff could make the value

$$\sum_{\{u,v\} \in E} \|\psi_i(u) - \psi_i(v)\|^2,$$

much larger than the corresponding quantity for  $F$ . Thus we must pursue a smoother approach for localizing  $F$ .

**The radial projection distance.** Our method of smooth localization depends crucially on defining a proper notion of distance between vertices, based on the map  $F$ . We would like to think of

two vertices  $u, v \in V$  as close if their Euclidean distance  $\|F(u) - F(v)\|$  is small compared to their norms  $\|F(u)\|, \|F(v)\|$ . To capture this, we define the *radial projection distance* via,

$$d_F(u, v) = \left\| \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right\|.$$

Note that a ball in  $d_F$  corresponds to a cone in  $\mathbb{R}^k$ ; see Figure 1.

Our goal now becomes to find separated regions  $S_1, \dots, S_k \subseteq V$  in  $d_F$ , each of which contains a large fraction of the  $\ell^2$  mass of  $F$ . If these regions are far enough apart, then there is a way to allow  $\psi_i$  to degrade gracefully off of  $S_i$ , ensuring that  $\mathcal{R}_G(\psi_i)$  remains small; see Lemma 3.3.

The isotropy condition (7) gives us the following energy spreading property of  $d_F$ : If  $S \subseteq V$ , then

$$\text{diam}(S, d_F) \leq \frac{1}{2} \implies \sum_{v \in S} \|F(v)\|^2 \leq \frac{2}{k} \sum_{v \in V} \|F(v)\|^2. \quad (8)$$

In other words, sets of small  $d_F$ -diameter cannot contain a large fraction of the  $\ell^2$  mass. This will be essential in finding regions  $\{S_i\}$ .

**Finding separated regions: Random space partitions.** In order to find many separated regions, we rely on the theory of random partitions discussed in Section 2.3. Roughly speaking, this partitions  $\mathbb{R}^k$  (and thus our set of points) randomly into pieces of diameter at most  $1/2$  so that the expected fraction of  $\ell^2$  mass which is close to the boundary of the partition is small. Thus we can take unions of the interiors of the pieces to find separated sets. Furthermore, no set in the partition can contain a large fraction of the  $\ell^2$  mass, due to the spreading property of  $d_F$  (8). This is carried out in Section 3.3. We use these separated sets as the supports of our family  $\{\psi_i\}$ , allowing us to complete the proof of Theorem 1.3.

The notion of “close to the boundary” depends on the dimension  $k$ , and thus the smoothness of our maps  $\{\psi_i\}$  will degrade as the dimension grows. For many families of graphs, however, we can appeal to special properties of their intrinsic geometry.

**Exploiting the intrinsic geometry.** It is well-known that the shortest-path metric on a planar graph has many nice properties, but  $d_F$  is, in general, not a shortest-path geometry. Thus it is initially unclear how one might prove a bound like (4) using our approach. The answer is to combine information from the spectral embedding with the intrinsic geometry of the graph.

We define  $\hat{d}_F$  as the shortest-path pseudometric on  $G$ , where the length of an edge  $\{u, v\} \in E$  is precisely  $d_F(u, v)$ . In Sections 3.2 and 3.3, we show that it is possible to do the partitioning in the metric  $\hat{d}_F$ , and thus for planar graphs (and other generalizations), we are able to achieve dimension-independent bounds in Theorem 1.2.

This technique also addresses a common shortcoming of spectral methods: The spectral embedding can lose auxiliary information about the input data that could help with clustering. Our “hybrid” technique for planar graphs suggests that such information (in this case, planarity) can be fruitfully combined with the spectral computations.

**Dimension reduction.** In order to obtain the tight bound (3) for general graphs, we have to improve the quantitative parameters of our construction significantly. The main loss in our preceding construction comes from the ambient dimension  $k$ .

Thus our first step is to apply dimension-reduction techniques: We randomly project our points from  $\mathbb{R}^k$  into  $\mathbb{R}^{O(\log k)}$ . Let  $F' : V \rightarrow \mathbb{R}^{O(\log k)}$  be the resulting map. While it is easy to see that

$\mathcal{R}_G(F') \asymp \mathcal{R}_G(F)$  with high probability, it is not, a priori, clear why  $O(\log k)$  dimensions suffices for maintaining the energy spreading properties of  $F$ . Indeed, the isotropy condition (7) will generally fail for  $F'$ . Although the proof is delicate (see Lemma 4.3), the basic idea is this: If  $d_F$  satisfies (8), but  $d_{F'}$  fails to satisfy a related property, then a  $\gg \frac{1}{k}$  fraction of the  $\ell^2$  mass has to have moved significantly in the dimension reduction step, and such an event is unlikely for a random mapping into  $O(\log k)$  dimensions.

**A new multi-way Cheeger inequality.** Dimension reduction only yields a loss of  $O(\log k)$  in (3). In order to get the bound down to  $\sqrt{\log k}$ , we have to abandon our goal of localizing eigenfunctions. In Section 4.2, we give a new multi-way Cheeger rounding algorithm that combines random partitions of the radial projection distance  $d_F$ , and random thresholding based on  $\|F(\cdot)\|$  (as in Cheeger's inequality). By analyzing these two processes simultaneously, we are able to achieve the optimal loss.

## 1.2 A general algorithm

Given a graph  $G = (V, E)$  and any embedding  $F : V \rightarrow \mathbb{R}^k$  (in particular, the spectral embedding (6)), our approach yields a general algorithmic paradigm for finding many non-expanding sets. For some  $r \in \mathbb{N}$ , do the following:

i) **(Radial decomposition)**

Find disjoint subsets  $S_1, S_2, \dots, S_r \subseteq V$  using the values  $\{F(v)/\|F(v)\| : v \in V\}$ .

ii) **(Cheeger sweep)**

For each  $i = 1, 2, \dots, r$ ,

Sort the vertices  $S_i = \{v_1, v_2, \dots, v_{n_i}\}$  so that

$$\|F(v_1)\| \leq \|F(v_2)\| \leq \dots \leq \|F(v_{n_i})\|.$$

Output the least-expanding set among the  $n_i - 1$  sets of the form,

$$\{v_1, v_2, \dots, v_j\}$$

for  $1 \leq j \leq n_i - 1$ .

As discussed in the preceding section, each of our main theorems is proved using an instantiation of this schema. For instance, the proof of Theorem 1.1 partitions using the radial projection distance  $d_F$ . The proof of (4) uses the induced shortest-path metric  $\hat{d}_F$ . And the proof of (3) uses  $d_{F'}$  where  $F' : V \rightarrow \mathbb{R}^{O(\log k)}$  is obtained from random projection. The details of the scheme for equation (3) is provided in Section 5. A practical algorithm might use  $r$ -means to cluster according to the radial projection distance.

We remark that partitioning the normalized vectors as in step (i) is used in the approach of [NJW02], but not in some other methods of spectral partitioning (see [VM03] for alternatives). Our analysis suggests a theoretical justification for partitioning using the normalized vectors.

## 2 Preliminaries

Let  $G = (V, E, w)$  be a finite, undirected graph, with positive weights  $w : E \rightarrow (0, \infty)$  on the edges. For a pair of vertices  $u, v \in V$ , we sometimes write  $w(u, v)$  for  $w(\{u, v\})$ . For a subset of vertices  $S \subseteq V$ , we write  $E(S, \bar{S}) := \{\{u, v\} \in E : |\{u, v\} \cap S| = 1\}$ . For a subset of edges  $F \subseteq E$ , we write  $w(F) = \sum_{e \in F} w(e)$ . We use  $x \sim y$  to denote  $\{x, y\} \in E$ . We extend the weight to vertices by defining, for a single vertex  $v \in V$ ,  $w(v) := \sum_{u \sim v} w(u, v)$ . We can think of  $w(v)$  as the weighted degree of vertex  $v$ . We will assume throughout that  $w(v) > 0$  for every  $v \in V$ . For  $S \subseteq V$ , we write  $w(S) = \sum_{v \in S} w(v)$ .

Let  $X$  be a set and  $d : X \times X \rightarrow [0, \infty]$  is a symmetric non-negative function which may take the value  $\infty$ . We refer to  $d$  as an *extended pseudo-metric on  $X$*  if it satisfies the triangle inequality. For a subset  $S \subseteq X$ , we write  $\text{diam}(S, d) := \sup_{x, y \in S} d(x, y)$ , and for two sets  $S, T \subseteq X$ , we write  $d(S, T) := \inf_{x \in S, y \in T} d(x, y)$ . We also define the ball  $B_d(x, R) := \{y \in X : d(x, y) \leq R\}$ .

For two expressions  $A$  and  $B$ , we write  $A \lesssim B$  for  $A \leq O(B)$  and  $A \asymp B$  for the conjunction of  $A \lesssim B$  and  $A \gtrsim B$ .

### 2.1 Spectral theory of the weighted Laplacian

We write  $\ell^2(V, w)$  for the Hilbert space of functions  $f : V \rightarrow \mathbb{R}$  with inner product

$$\langle f, g \rangle_{\ell^2(V, w)} := \sum_{v \in V} w(v) f(v) g(v),$$

and norm  $\|f\|_{\ell^2(V, w)}^2 = \langle f, f \rangle_{\ell^2(V, w)}$ . We reserve  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  for the standard inner product and norm on  $\mathbb{R}^k$ ,  $k \in \mathbb{N}$  and  $\ell^2(V)$ .

We now discuss some operators on  $\ell^2(V, w)$ . The adjacency operator is defined by  $Af(v) = \sum_{u \sim v} w(u, v) f(u)$ , and the diagonal degree operator by  $Df(v) = w(v) f(v)$ . Then the *combinatorial Laplacian* is defined by  $L = D - A$ , and the *normalized Laplacian* is given by

$$\mathcal{L}_G := I - D^{-1/2} A D^{-1/2}.$$

Observe that for an unweighted,  $d$ -regular graph, we have  $\mathcal{L}_G = \frac{1}{d} L$ .

Now, if  $g : V \rightarrow \mathbb{R}$  is a non-zero function and  $f = D^{-1/2} g$ , then

$$\begin{aligned} \frac{\langle g, \mathcal{L}_G g \rangle}{\langle g, g \rangle} &= \frac{\langle g, D^{-1/2} L D^{-1/2} g \rangle}{\langle g, g \rangle} \\ &= \frac{\langle f, L f \rangle}{\langle D^{1/2} f, D^{1/2} f \rangle} \\ &= \frac{\sum_{u \sim v} w(u, v) |f(u) - f(v)|^2}{\sum_{v \in V} w(v) f(v)^2} =: \mathcal{R}_G(f), \end{aligned}$$

where the latter value is referred to as the *Rayleigh quotient of  $f$  (with respect to  $G$ )*.

In particular, one sees that  $\mathcal{L}_G$  is a positive-definite operator with eigenvalues

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq 2.$$



For a connected graph, the first eigenvalue corresponds to the eigenfunctions  $g = D^{1/2}f$ , where  $f$  is any non-zero constant function. Furthermore, by standard variational principles,

$$\begin{aligned}\lambda_k &= \min_{g_1, \dots, g_k \in \ell^2(V)} \max_{g \neq 0} \left\{ \frac{\langle g, \mathcal{L}_G g \rangle}{\langle g, g \rangle} : g \in \text{span}\{g_1, \dots, g_k\} \right\} \\ &= \min_{f_1, \dots, f_k \in \ell^2(V, w)} \max_{f \neq 0} \left\{ \mathcal{R}_G(f) : f \in \text{span}\{f_1, \dots, f_k\} \right\},\end{aligned}\tag{9}$$

where both minimums are over sets of  $k$  non-zero orthogonal functions in the Hilbert spaces  $\ell^2(V)$  and  $\ell^2(V, w)$ , respectively. We refer to [Chu97] for more background on the spectral theory of the normalized Laplacian.

## 2.2 Cheeger's inequality with Dirichlet boundary conditions

Given a subset  $S \subseteq V$  by, we denote the *Dirichlet conductance* of  $S$  by,

$$\phi_G(S) := \frac{w(E(S, \bar{S}))}{w(S)}.$$

If  $\mathcal{H}$  is a Hilbert space, we extend the notion of Rayleigh quotients to arbitrary maps  $\psi : V \rightarrow \mathcal{H}$  via,

$$\mathcal{R}_G(\psi) := \frac{\sum_{u \sim v} w(u, v) \|\psi(u) - \psi(v)\|_{\mathcal{H}}^2}{\sum_{v \in V} w(v) \|\psi(v)\|_{\mathcal{H}}^2}.\tag{10}$$

In what follows, we use  $\text{supp}(\psi) := \{v \in V : \psi(v) \neq 0\}$ .

Many variants of the following lemma are known; see, e.g. [Chu96].

**Lemma 2.1.** *For any  $\psi : V \rightarrow \mathcal{H}$ , there exists a subset  $S \subseteq \text{supp}(\psi)$  with*

$$\phi_G(S) \leq \sqrt{2 \mathcal{R}_G(\psi)}.$$

*Proof.* Let  $\|\cdot\| = \|\cdot\|_{\mathcal{H}}$ . We may assume that  $\text{supp}(\psi) \neq V$ , else taking  $S = V$  finishes the argument. Since  $\mathcal{R}_G(\psi)$  is homogeneous in  $\psi$ , we may assume that  $\psi : V \rightarrow [-1, 1]$ . Define a subset  $S_t = \{u \in V : \|\psi(u)\|^2 \geq t\}$ , and let  $t \in (0, 1]$  be chosen uniformly at random. Observe that  $S_t \subseteq \text{supp}(\psi)$  by construction.

Then we have the estimate,

$$\mathbb{E}[w(S_t)] = \sum_{u \in V} w(u) \|\psi(u)\|^2,$$

as well as,

$$\begin{aligned}\mathbb{E}[w(E(S_t, \bar{S}_t))] &= \sum_{u \sim v} w(u, v) \left| \|\psi(u)\|^2 - \|\psi(v)\|^2 \right| \\ &\leq \sum_{u \sim v} w(u, v) \|\psi(u) - \psi(v)\| \cdot \|\psi(u) + \psi(v)\| \\ &\leq \sqrt{\sum_{u \sim v} w(u, v) \|\psi(u) - \psi(v)\|^2} \sqrt{\sum_{u \sim v} w(u, v) \|\psi(u) + \psi(v)\|^2} \\ &\leq \sqrt{\sum_{u \sim v} w(u, v) \|\psi(u) - \psi(v)\|^2} \sqrt{2 \sum_{u \in V} w(u) \|\psi(u)\|^2}.\end{aligned}$$

Combining these two inequalities yields,

$$\frac{\mathbb{E}[w(E(S_t, \overline{S_t}))]}{\mathbb{E}[w(S_t)]} \leq \sqrt{2\mathcal{R}(\psi)},$$

implying there exists a  $t \in [0, 1]$  for which  $S_t$  satisfies the statement of the lemma.  $\square$

### 2.3 Random partitions of metric spaces

We now discuss some of the theory of random partitions of metric spaces. Let  $(X, d)$  be a finite metric space. We use  $B(x, R) = \{y \in X : d(x, y) \leq R\}$  to denote the closed ball of radius  $R$  about  $x$ . We will write a partition  $P$  of  $X$  as a function  $P : X \rightarrow 2^X$  mapping a point  $x \in X$  to the unique set in  $P$  that contains  $x$ .

For  $\Delta > 0$ , we say that  $P$  is  $\Delta$ -*bounded* if  $\text{diam}(S) \leq \Delta$  for every  $S \in P$ . We will also consider distributions over random partitions. If  $\mathcal{P}$  is a random partition of  $X$ , we say that  $\mathcal{P}$  is  $\Delta$ -bounded if this property holds with probability one.

A random partition  $\mathcal{P}$  is  $(\Delta, \alpha, \delta)$ -*padded* if  $\mathcal{P}$  is  $\Delta$ -bounded, and for every  $x \in X$ , we have

$$\mathbb{P}[B(x, \Delta/\alpha) \subseteq \mathcal{P}(x)] \geq \delta.$$

A random partition is  $(\Delta, L)$ -*Lipschitz* if  $\mathcal{P}$  is  $\Delta$ -bounded, and, for every pair  $x, y \in X$ , we have

$$\mathbb{P}[\mathcal{P}(x) \neq \mathcal{P}(y)] \leq L \cdot \frac{d(x, y)}{\Delta}.$$

Here are some results that we will need. The first theorem is known, more generally, for doubling spaces [GKL03], but here we only need its application to  $\mathbb{R}^k$ . See also [LN05, Lem 3.11].

**Theorem 2.2.** *If  $X \subseteq \mathbb{R}^k$ , then for every  $\Delta > 0$  and  $\delta > 0$ ,  $X$  admits a  $(\Delta, O(k/\delta), 1 - \delta)$ -padded random partition.*

The next result is proved in [CCG<sup>+</sup>98]. See also [LN05, Lem 3.16].

**Theorem 2.3.** *If  $X \subseteq \mathbb{R}^k$ , then for every  $\Delta > 0$ ,  $X$  admits a  $(\Delta, O(\sqrt{k}))$ -Lipschitz random partition.*

A partitioning theorem for excluded-minor graphs is presented in [KPR93], with an improved quantitative dependence coming from [FT03].

**Theorem 2.4.** *If  $X$  is the shortest-path metric on a graph excluding  $K_h$  as a minor, then for every  $\Delta > 0$  and  $\delta > 0$ ,  $X$  admits a  $(\Delta, O(h^2/\delta), 1 - \delta)$ -padded random partition and a  $(\Delta, O(h^2))$ -Lipschitz random partition.*

Finally, for the special case of bounded-genus graphs, a better bound is known [LS10].

**Theorem 2.5.** *If  $X$  is the shortest-path metric on a graph of genus  $g$ , for every  $\Delta > 0$  and  $\delta > 0$ ,  $X$  admits a  $(\Delta, O((\log g)/\delta), 1 - \delta)$ -padded random partition, and a  $(\Delta, O(\log g))$ -Lipschitz random partition.*

### 3 Localizing eigenfunctions

Let  $G = (V, E, w)$  be a weighted graph. In the present section, we show how to find, for every  $k \in \mathbb{N}$ , disjointly supported functions  $\psi_1, \psi_2, \dots, \psi_k : V \rightarrow \mathbb{R}$  with  $\mathcal{R}_G(\psi_i) \leq k^{O(1)} \lambda_k$ , where  $\lambda_k$  is the  $k$ th smallest eigenvalue of  $\mathcal{L}_G$ .

#### 3.1 The radial projection distance

For  $h \in \mathbb{N}$ , consider a mapping  $F : V \rightarrow \mathbb{R}^h$ . A central role will be played by the *radial projection distance*, which is an extended pseudo-metric on  $V$ : If  $\|F(u)\|, \|F(v)\| > 0$ , then

$$d_F(u, v) := \left\| \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right\|.$$

Otherwise, if  $F(u) = F(v) = 0$ , we put  $d_F(u, v) := 0$ , else  $d_F(u, v) := \infty$ .

In order to find many disjointly supported functions from a geometric representation  $F : V \rightarrow \mathbb{R}^h$ , it should be that the  $\ell^2$  mass of  $F$  is not too concentrated. To this end, we say that  $F$  is  $(\Delta, \eta)$ -*spreading* (with respect to  $G$ ) if, for all subsets  $S \subseteq V$ , we have

$$\text{diam}(S, d_F) \leq \Delta \implies \sum_{u \in S} w(u) \|F(u)\|^2 \leq \eta \sum_{u \in V} w(u) \|F(u)\|^2.$$

First, we record the following simple fact.

**Lemma 3.1.** *For any  $F : V \rightarrow \mathbb{R}^h$ , and for all  $u, v \in V$ , we have  $d_F(u, v) \|F(u)\| \leq 2 \|F(u) - F(v)\|$ .*

*Proof.* For any non-zero vectors  $x, y \in \mathbb{R}^k$ , we have

$$\|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| = \left\| x - \frac{\|x\|}{\|y\|} y \right\| \leq \|x - y\| + \left\| y - \frac{\|x\|}{\|y\|} y \right\| \leq 2 \|x - y\|.$$

□

We now show that systems of  $\ell^2(V, w)$ -orthonormal functions give rise to spreading maps.

**Lemma 3.2.** *Suppose that  $f_1, f_2, \dots, f_k : V \rightarrow \mathbb{R}$  is an  $\ell^2(V, w)$ -orthonormal system and that  $F : V \rightarrow \mathbb{R}^k$  is given by  $F(v) = (f_1(v), f_2(v), \dots, f_k(v))$ . Then, for every  $\Delta > 0$ ,  $F$  is  $(\Delta, \frac{1}{k(1-\Delta^2)})$ -spreading with respect to  $G$ .*

*Proof.* Let  $x \in \mathbb{R}^k$  be any unit vector, and let  $U : \mathbb{R}^k \rightarrow \ell^2(V, w)$  be defined by

$$(Ux)(v) := \sum_{i=1}^k x_i \sqrt{w(v)} f_i(v).$$

Observe that  $(U^T U)_{i,j} = \langle f_i, f_j \rangle_{\ell^2(V, w)}$ , hence  $U^T U = I$ . Thus,

$$\sum_{v \in V} w(v) \langle x, F(v) \rangle^2 = \langle Ux, Ux \rangle = \langle x, U^T Ux \rangle = 1. \quad (11)$$

Now, let  $S \subseteq V$  satisfy  $\text{diam}(S, d_F) \leq \Delta$ . Fix any  $u \in S$  and use (11) to write,

$$1 = \sum_{v \in V} w(v) \left\langle F(v), \frac{F(u)}{\|F(u)\|} \right\rangle^2 = \sum_{v \in V} w(v) \|F(v)\|^2 \left( 1 - \frac{d_F(u, v)^2}{2} \right)^2 \geq (1 - \Delta^2) \sum_{v \in S} w(v) \|F(v)\|^2.$$

The lemma now follows by noting that,

$$\sum_{v \in V} w(v) \|F(v)\|^2 = \sum_{v \in V} \sum_{i=1}^k w(v) f_i(v)^2 = \sum_{v \in V} \sum_{i=1}^k \|f_i\|_{\ell^2(V, w)}^2 = k.$$

□

### 3.2 Smooth localization

Given a map  $F : V \rightarrow \mathbb{R}^h$  and a subset  $S \subseteq V$ , we now show how to construct a function supported on a small-neighborhood  $S$ , which retains the  $\ell^2$  mass of  $F$  on  $S$ , and which doesn't stretch edges by too much.

For future applications, it will be useful to consider the largest metric on  $G$  which agrees with  $d_F$  on edges. This is the induced shortest-path (extended pseudo-) metric on  $G$ , where the length of an edge  $\{u, v\} \in E$  is given by  $d_F(u, v)$ . We will use the notation  $\hat{d}_F$  for this metric. Observe that  $\hat{d}_F \geq d_F$  since  $d_F$  is a pseudo-metric. We will write

$$N_\varepsilon(S, \hat{d}_F) := \{v \in V : \hat{d}_F(v, S) < \varepsilon\}$$

for the open  $\varepsilon$ -neighborhood of  $S$  in the metric  $\hat{d}_F$ .

**Lemma 3.3** (Localization). *For any  $F : V \rightarrow \mathbb{R}^h$ , the following holds. For every subset  $S \subseteq V$  and number  $\varepsilon > 0$ , there exists a mapping  $\psi : V \rightarrow \mathbb{R}^h$  which satisfies the following three properties:*

- i)  $\psi|_S = F|_S$ ,
- ii)  $\text{supp}(\psi) \subseteq N_\varepsilon(S, \hat{d}_F)$ , and
- iii) if  $\{u, v\} \in E$ , then  $|\psi(u) - \psi(v)| \leq (1 + \frac{2}{\varepsilon}) \|F(u) - F(v)\|$ .

*Proof.* First, define

$$\theta(v) := \max \left( 0, 1 - \frac{\hat{d}_F(v, S)}{\varepsilon} \right).$$

In particular, observe that  $\theta$  is  $(1/\varepsilon)$ -Lipschitz with respect to  $\hat{d}_F$ , so since  $\hat{d}_F$  and  $d_F$  agree on edges, we have for every  $\{u, v\} \in E$ ,

$$|\theta(u) - \theta(v)| \leq \frac{1}{\varepsilon} d_F(u, v). \tag{12}$$

Finally, set  $\psi(v) := \theta(v)F(v)$ .

Properties (i) and (ii) are immediate from the definition, thus we turn to property (iii). Fix  $\{u, v\} \in E$ . We have,

$$\begin{aligned} |\psi(u) - \psi(v)| &= |\theta(u)F(u) - \theta(v)F(v)| \\ &\leq |\theta(v)| \cdot \|F(u) - F(v)\| + \|F(u)\| \cdot |\theta(u) - \theta(v)|. \end{aligned}$$

Since  $\theta \leq 1$ , the first term is at most  $\|F(u) - F(v)\|$ . Now, using (12), and Lemma 3.1, we have

$$\|F(u)\| \cdot |\theta(u) - \theta(v)| \leq \frac{1}{\varepsilon} \cdot \|F(u)\| \cdot d_F(u, v) \leq \frac{2}{\varepsilon} \cdot \|F(u) - F(v)\|,$$

completing the proof of (iii).  $\square$

The preceding construction reduces the problem of finding disjointly supported set functions to finding separated regions in  $(V, \hat{d}_F)$ , each of which contains a large fraction of the  $\ell^2$  mass of  $F$ .

**Lemma 3.4.** *Let  $F : V \rightarrow \mathbb{R}^h$  be given, and suppose that for some  $\beta, \delta > 0$  and  $r \in \mathbb{N}$ , there exist  $r$  disjoint subsets  $T_1, T_2, \dots, T_r \subseteq V$  such that  $\hat{d}_F(T_i, T_j) \geq \beta$  for  $i \neq j$ , and for every  $i = 1, 2, \dots, r$ , we have*

$$\sum_{v \in T_i} w(v) \|F(v)\|^2 \geq \delta \sum_{v \in V} w(v) \|F(v)\|^2. \quad (13)$$

*Then there exist disjointly supported functions  $\psi_1, \psi_2, \dots, \psi_r : V \rightarrow \mathbb{R}$  such that for  $i = 1, 2, \dots, r$ , we have*

$$\mathcal{R}_G(\psi_i) \leq \frac{2}{\delta(r-i+1)} \left(1 + \frac{4}{\beta}\right)^2 \mathcal{R}_G(F). \quad (14)$$

*Proof.* For each  $i \in [r]$ , let  $\psi_i : V \rightarrow \mathbb{R}^h$  be the result of applying Lemma 3.3 to the domain  $T_i$  with parameter  $\varepsilon = \beta/2$ . Since  $\hat{d}_F(T_i, T_j) \geq \beta$  for  $i \neq j$ , property (ii) of Lemma 3.3 ensures that the functions  $\{\psi_i\}_{i=1}^r$  are disjointly supported.

Additionally property (i) implies that for each  $i \in [r]$ ,

$$\sum_{v \in V} w(v) \|\psi_i(v)\|^2 \geq \sum_{v \in T_i} w(v) \|F(v)\|^2 \geq \delta \sum_{v \in V} w(v) \|F(v)\|^2,$$

and by property (iii) of Lemma 3.3, and since the supports are disjoint,

$$\sum_{u \sim v} \sum_{i=1}^r w(u, v) \|\psi_i(u) - \psi_i(v)\|^2 \leq 2 \left(1 + \frac{4}{\beta}\right)^2 \sum_{u \sim v} w(u, v) \|F(u) - F(v)\|^2.$$

In particular, if we reorder the maps so that  $\mathcal{R}_G(\psi_1) \leq \mathcal{R}_G(\psi_2) \leq \dots \leq \mathcal{R}_G(\psi_r)$ , then the preceding two inequalities imply (14).

These maps  $\{\psi_i\}$  take values in  $\mathbb{R}^h$ , but it is easy to see that for any  $\psi : V \rightarrow \mathbb{R}^h$ , there exists a coordinate  $j \in \{1, 2, \dots, h\}$  such that the map  $\tilde{\psi} : V \rightarrow \mathbb{R}$  defined by  $\tilde{\psi}(v) := \psi(v)_j$  has  $\mathcal{R}_G(\tilde{\psi}) \leq \mathcal{R}_G(\psi)$ . This follows from the general inequality  $\frac{a_1 + a_2 + \dots + a_k}{b_1 + b_2 + \dots + b_k} \geq \min_i \frac{a_i}{b_i}$ , valid for all  $a_1, \dots, a_k, b_1, \dots, b_k \geq 0$  with some  $b_i > 0$ .  $\square$

### 3.3 Random partitioning

From Lemma 3.4, to find many disjointly supported functions with small Rayleigh quotient, it suffices to partition  $(V, \hat{d}_F)$  into well separated regions, each of which contains a large fraction of the  $\ell^2$  mass of  $F$ . We will use a suitable distribution over random partitions and argue that at least one partition in the support of the distribution is good for this purpose.

**Lemma 3.5.** *Let  $r, k \in \mathbb{N}$  be given with  $k/2 \leq r \leq k$ , and suppose that the map  $F : V \rightarrow \mathbb{R}^h$  is  $(\Delta, \frac{1}{k} + \frac{k-r+1}{8kr})$ -spreading for some  $\Delta > 0$ . Suppose additionally there is a random partition  $\mathcal{P}$  with the properties that*

i) For every  $S \in \mathcal{P}$ ,  $\text{diam}(S, d_F) \leq \Delta$ , and

ii) For every  $v \in V$ ,  $\mathbb{P}[B_{\hat{d}_F}(v, \Delta/\alpha) \subseteq \mathcal{P}(v)] \geq 1 - \frac{k-r+1}{4r}$ .

Then there exist  $r$  disjoint subsets  $T_1, T_2, \dots, T_r \subseteq V$  such that for each  $i \neq j$ , we have  $\hat{d}_F(T_i, T_j) \geq 2\Delta/\alpha$ , and for every  $i = 1, 2, \dots, k$ ,

$$\sum_{v \in T_i} w(v) \|F(v)\|^2 \geq \frac{1}{2k} \sum_{v \in V} w(v) \|F(v)\|^2.$$

*Proof.* For a subset  $S \subseteq V$ , define

$$\tilde{S} := \{x \in S : B_{\hat{d}_F}(x, \Delta/\alpha) \subseteq S\}.$$

Let  $\mathcal{E} = \sum_{v \in V} w(v) \|F(v)\|^2 > 0$ . By linearity of expectation, there exists a partition  $P$  such that for every  $S \in P$ ,  $\text{diam}(S, d_F) \leq \Delta$ , and also

$$\sum_{S \in P} \sum_{v \in \tilde{S}} w(v) \|F(v)\|^2 \geq \left(1 - \frac{k-r+1}{4r}\right) \mathcal{E}. \quad (15)$$

Furthermore, by the spreading property of  $F$ , we have, for each  $S \in P$ ,

$$\sum_{x \in S} w(v) \|F(v)\|^2 \leq \frac{1}{k} \left(1 + \frac{k-r+1}{8r}\right) \mathcal{E}.$$

Therefore we may take disjoint unions of the sets  $\{\tilde{S} : S \in P\}$  to form at least  $r$  disjoint sets  $T_1, T_2, \dots, T_r$  with the property that for every  $i = 1, 2, \dots, r$ , we have

$$\sum_{v \in T_i} w(v) \|F(v)\|^2 \geq \frac{1}{2k} \mathcal{E}$$

because the first  $r-1$  pieces will have total mass at most

$$\frac{r-1}{k} \left(1 + \frac{k-r+1}{8r}\right) \mathcal{E} \leq \left(1 - \frac{k-r+1}{4r} - \frac{1}{2k}\right) \mathcal{E},$$

for all  $r \in [k/2, k]$ , leaving at least  $\frac{\mathcal{E}}{2k}$  mass left over from (15).  $\square$

We mention a representative corollary that follows from the conjunction of Lemmas 3.4 and 3.5.

**Corollary 3.6.** *Let  $k \in \mathbb{N}$  and  $\delta \in (0, 1)$  be given. Suppose the map  $F : V \rightarrow \mathbb{R}^h$  is  $(\Delta, \frac{1}{k} + \frac{\delta}{48k})$ -spreading for some  $\Delta \leq 1$ , and there is a random partition  $\mathcal{P}$  with the properties that*

i) For every  $S \in \mathcal{P}$ ,  $\text{diam}(S, d_F) \leq \Delta$ , and

ii) For every  $v \in V$ ,  $\mathbb{P}[B_{\hat{d}_F}(v, \Delta/\alpha) \subseteq \mathcal{P}(v)] \geq 1 - \frac{\delta}{24}$ .

Then there are at least  $r \geq \lceil (1 - \delta)k \rceil$  disjointly supported functions  $\psi_1, \psi_2, \dots, \psi_r : V \rightarrow \mathbb{R}$  such that

$$\mathcal{R}_G(\psi_i) \lesssim \frac{\alpha^2}{\delta \Delta^2} \mathcal{R}_G(F).$$

*Proof.* In this case, we set  $r = \lceil (1 - \delta/2)k \rceil$  in our application of Lemma 3.5. After extracting at least  $\lceil (1 - \delta/2)k \rceil$  sets, we apply Lemma 3.4, but only take the first  $r' = \lceil (1 - \delta)k \rceil$  functions  $\psi_1, \psi_2, \dots, \psi_{r'}$ .  $\square$

Note, in particular, that we can apply the preceding corollary with  $\delta = \frac{1}{2k}$  to obtain  $r = k$ .

### 3.4 Higher-order Cheeger inequalities

We now present some theorems applying our machinery to embeddings which come from the eigenfunctions of  $\mathcal{L}_G$ .

**Theorem 3.7.** *For any  $\delta \in (0, 1)$ , and any weighted graph  $G = (V, E, w)$ , there exist  $r \geq \lceil (1 - \delta)k \rceil$  disjointly supported functions  $\psi_1, \psi_2, \dots, \psi_r : V \rightarrow \mathbb{R}$  such that*

$$\mathcal{R}_G(\psi_i) \lesssim \frac{k^2}{\delta^4} \lambda_k. \quad (16)$$

where  $\lambda_k$  is the  $k$ th smallest eigenvalue of  $\mathcal{L}_G$ . If  $G$  excludes  $K_h$  as a minor, then the bound improves to

$$\mathcal{R}_G(\psi_i) \lesssim \frac{h^4}{\delta^4} \lambda_k, \quad (17)$$

and if  $G$  has genus at most  $g \geq 1$ , then one gets

$$\mathcal{R}_G(\psi_i) \lesssim \frac{\log^2(g + 1)}{\delta^4} \lambda_k. \quad (18)$$

*Proof.* Let  $f_1, f_2, \dots, f_k : V \rightarrow \mathbb{R}$  be an  $\ell^2(V, w)$ -orthonormal system of eigenfunctions corresponding to the first  $k$  eigenvalues of  $\mathcal{L}_G$ , and define  $F : V \rightarrow \mathbb{R}^k$  by  $F(v) = (f_1(v), f_2(v), \dots, f_k(v))$ .

Choose  $\Delta \asymp \sqrt{\delta}$  so that  $(1 - \Delta^2)^{-1} \leq 1 + \frac{\delta}{48}$ . In this case, Lemma 3.2 implies that  $F$  is  $(\Delta, \frac{1}{k} + \frac{\delta}{48k})$ -spreading. Now, for general graphs, since  $d_F$  is Euclidean, we can use Theorem 2.2 applied to  $d_F$  to achieve  $\alpha \asymp k/\delta$  in the assumptions of Corollary 3.6. Observe that  $\hat{d}_F \geq d_F$ , so that  $B_{\hat{d}_F}(v, \Delta/\alpha) \subseteq B_{d_F}(v, \Delta/\alpha)$ , meaning that we can satisfy both conditions (i) and (ii), verifying (16).

For (17) and (18), we use Theorems 2.4 and 2.5, respectively, applied to the shortest-path metric  $\hat{d}_F$ . Again, since  $\hat{d}_F \geq d_F$ , we have that  $\text{diam}(S, \hat{d}_F) \leq \Delta$  implies  $\text{diam}(S, d_F) \leq \Delta$ , so conditions (i) and (ii) are satisfied with  $\alpha \asymp h^2/\delta$  and  $\alpha \asymp \log(g + 1)/\delta$ , respectively.  $\square$

We remark that in Section 4.1, we will give an alternate bound of  $O(\delta^{-7} \log^2 k) \cdot \lambda_k$  for (16), which is better for moderate values of  $\delta$ .

Finally, we can use the preceding theorems in conjunction with Lemma 2.1 to produce many non-expanding sets.

**Theorem 3.8.** *(Non-expanding  $k$ -partition) For any weighted graph  $G = (V, E, w)$ , there exists a partition  $V = S_1 \cup S_2 \cup \dots \cup S_k$  such that*

$$\phi_G(S_i) \lesssim k^4 \sqrt{\lambda_k}.$$

where  $\lambda_k$  is the  $k$ th smallest eigenvalue of  $\mathcal{L}_G$ . If  $G$  excludes  $K_h$  as a minor, then the bound improves to

$$\phi_G(S_i) \lesssim h^2 k^3 \sqrt{\lambda_k},$$

and if  $G$  has genus at most  $g \geq 1$ , then one gets

$$\phi_G(S_i) \lesssim \log(g + 1) k^3 \sqrt{\lambda_k}.$$

*Proof.* First apply Theorem 3.7 with  $\delta = \frac{1}{2k}$  to find disjointly supported functions  $\psi_1, \psi_2, \dots, \psi_k : V \rightarrow \mathbb{R}$  satisfying (16). Now apply Lemma 2.1 to find sets  $S_1, S_2, \dots, S_k$  with  $S_i \subseteq \text{supp}(\psi_i)$  and  $\phi_G(S_i) \leq \sqrt{2\mathcal{R}_G(\psi_i)}$  for each  $i = 1, 2, \dots, k$ .

Now reorder the sets so that  $w(S_1) \leq w(S_2) \leq \dots \leq w(S_k)$ , and replace  $S_k$  with the larger set  $S'_k = V \setminus (S_1 \cup S_2 \cup \dots \cup S_{k-1})$  so that  $V = S_1 \cup S_2 \cup \dots \cup S_{k-1} \cup S'_k$  forms a partition. One can now easily check that

$$\phi_G(S'_k) = \frac{w(E(S'_k, \overline{S'_k}))}{w(S'_k)} \leq \frac{\sum_{i=1}^{k-1} w(E(S_i, \overline{S_i}))}{w(S'_k)} \leq k \cdot \max_{i=1}^k \phi_G(S_i) \lesssim k^4 \sqrt{\lambda_k}.$$

A similar argument yields the other two bounds.  $\square$

Using Theorem 3.7 in conjunction with Lemma 2.1 again yields the following.

**Theorem 3.9.** *For every  $\delta \in (0, 1)$  and any weighted graph  $G = (V, E, w)$ , there exist  $r \geq \lceil (1-\delta)k \rceil$  disjoint sets  $S_1, S_2, \dots, S_r \subseteq V$  such that,*

$$\phi_G(S_i) \lesssim \frac{k}{\delta^2} \sqrt{\lambda_k}. \quad (19)$$

where  $\lambda_k$  is the  $k$ th smallest eigenvalue of  $\mathcal{L}_G$ . If  $G$  excludes  $K_h$  as a minor, then the bound improves to

$$\phi_G(S_i) \lesssim \frac{h^2}{\delta^2} \sqrt{\lambda_k},$$

and if  $G$  has genus at most  $g \geq 1$ , then one gets

$$\phi_G(S_i) \lesssim \frac{\log(g+1)}{\delta^2} \sqrt{\lambda_k}.$$

We remark that the bound (19) will be improved, in various ways, in Section 4.

## 4 Improved quantitative bounds

A main result of this section is the following theorem.

**Theorem 4.1.** *Let  $G = (V, E, w)$  be a weighted graph and let  $k \in \{1, 2, \dots, n\}$  and  $\delta \in (0, 1)$  be given. Suppose that  $f_1, f_2, \dots, f_k : V \rightarrow \mathbb{R}$  forms an  $\ell^2(V, w)$ -orthonormal system. Then there exist  $r \geq \lceil (1-\delta)k \rceil$  disjoint sets  $S_1, S_2, \dots, S_r \subseteq V$  with*

$$\phi_G(S_i) \lesssim \frac{1}{\delta^3} \sqrt{\frac{\sum_{i=1}^k \sum_{u \sim v} w(u, v) (f_i(u) - f_i(v))^2}{\sum_{i=1}^k \sum_{v \in V} w(v) f_i(v)^2} \cdot \log k}$$

**Corollary 4.2.** *For any weighted graph  $G = (V, E, w)$ ,  $k \in \{1, 2, \dots, n\}$ , and  $\delta \in (0, 1)$ , there exist  $r \geq \lceil (1-\delta)k \rceil$  disjoint sets  $S_1, S_2, \dots, S_r \subseteq V$  with*

$$\phi_G(S_i) \lesssim \frac{1}{\delta^3} \sqrt{\lambda_k \log k},$$

where  $\lambda_k$  is the  $k$ th smallest eigenvalue of  $\mathcal{L}_G$ .



## 4.1 Dimension reduction

One should observe that in Theorems 3.7 and 3.9, the loss of  $k^2$  in (16) and  $k$  in (19) comes from the dimension of the eigenfunction embedding. To achieve somewhat better bounds for general graphs, we now show how to drastically reduce the dimension while preserving the Rayleigh quotient and spreading properties.

Let  $g_1, g_2, \dots, g_h$  be i.i.d.  $k$ -dimensional Gaussians, and consider the random mapping  $\Gamma_{k,h} : \mathbb{R}^k \rightarrow \mathbb{R}^h$  defined by  $\Gamma_{k,h}(x) = h^{-1/2}(\langle g_1, x \rangle, \langle g_2, x \rangle, \dots, \langle g_h, x \rangle)$ . Then we have the following basic estimates (see, e.g. [Mat02, Ch. 15] or [LT11, Ch. 1]). For every  $x \in \mathbb{R}^k$ ,

$$\mathbb{E} [\|\Gamma_{k,h}(x)\|^2] = \|x\|^2, \quad (20)$$

and, for every  $\delta \in (0, \frac{1}{2}]$ ,

$$\mathbb{P} [\|\Gamma_{k,h}(x)\|^2 \notin [(1-\delta)\|x\|^2, (1+\delta)\|x\|^2]] \leq 2e^{-\delta^2 h/12}, \quad (21)$$

and for every  $\lambda \geq 2$ ,

$$\mathbb{P} [\|\Gamma_{k,h}(x)\|^2 \geq \lambda \|x\|^2] \leq e^{-\lambda h/12}. \quad (22)$$

**Lemma 4.3.** *Let  $G = (V, E, w)$  be a weighted graph. For every  $k \in \mathbb{N}$ ,  $\Delta \in [0, 1]$ , and  $\eta \geq 1/k$ , the following holds. Suppose that  $F : V \rightarrow \mathbb{R}^k$  is  $(\Delta, \eta)$ -spreading. Then for some value*

$$h \lesssim \frac{1 + \log(k) + \log(\frac{1}{\Delta})}{\Delta^2},$$

*with probability at least  $1/2$ , the map  $\Gamma_{k,h}$  satisfies both of the following conditions:*

- i)  $\mathcal{R}_G(\Gamma_{k,h} \circ F) \leq 8 \cdot \mathcal{R}_G(F)$ , and
- ii)  $\Gamma_{k,h} \circ F$  is  $(\Delta/4, (1+\Delta)\eta)$ -spreading with respect to  $G$ .

*Proof.* Let  $\delta = \Delta/16$ . We may assume that  $k \geq 2$ . Choose  $h \asymp (1 + \log k + \log(\frac{1}{\Delta}))/\Delta^2$  large enough such that  $2e^{-\delta^2 h/12} \leq \delta^2 k^{-3}/128$ . Let  $\Gamma = \Gamma_{k,h}$ .

First, observe that (20) combined with Markov's inequality implies that the following holds with probability at least  $3/4$ ,

$$\sum_{u \sim v} w(u, v) \|\Gamma(F(u)) - \Gamma(F(v))\|^2 \leq 4 \cdot \sum_{u \sim v} w(u, v) \|F(u) - F(v)\|^2. \quad (23)$$

Now define,

$$U := \{v \in V : \|\Gamma(F(v))\|^2 \in [(1-\delta)\|F(v)\|^2, (1+\delta)\|F(v)\|^2],$$

By (21), for each  $v \in V$ ,

$$\mathbb{P}[v \notin U] \leq \delta k^{-3}/128. \quad (24)$$

Next, we bound the amount of  $\ell^2$  mass that falls outside of  $U$ .

Therefore, by Markov's inequality, with probability at least  $31/32$ , we have

$$\sum_{v \notin U} w(v) \|F(v)\|^2 \leq \frac{\delta k^{-3}}{4} \sum_{v \in V} w(v) \|F(v)\|^2. \quad (25)$$

In particular, with probability at least  $31/32$ , we have

$$\sum_{v \in V} w(v) \|\Gamma(F(v))\|^2 \geq (1 - \delta) \sum_{v \in U} w(v) \|F(v)\|^2 \geq (1 - 2\delta) \sum_{v \in V} w(v) \|F(v)\|^2. \quad (26)$$

Combining our estimates for (23) and (26), we conclude that (i) holds with probability at least  $23/32$ . Thus we can finish by showing that (ii) holds with probability at least  $25/32$ . We first consider property (ii) for subsets of  $U$ .

**Claim 4.4.** *With probability at least  $7/8$ , the following holds: Equation (26) implies that, for any subset  $S \subseteq U$  with  $\text{diam}(S) \leq \Delta/4$ , we have*

$$\sum_{v \in S} w(v) \|\Gamma(F(v))\|^2 \leq (1 + 6\delta)\eta \sum_{v \in V} w(v) \|\Gamma(F(v))\|^2.$$

*Proof.* For every  $u, v \in V$ , define the event,

$$\mathcal{A}_{u,v} = \{d_{\Gamma(F)}(u, v) \in [d_F(u, v)(1 - \delta) - 2\delta, d_F(u, v)(1 + \delta) + 2\delta]\}$$

and let  $I_{u,v}$  be the random variable indicating that  $\mathcal{A}_{u,v}$  does *not* occur.

We claim that for  $u, v \in V$ ,  $\mathcal{A}_{u,v}$  occurs if  $u, v \in U$ , and

$$\left\| \Gamma \left( \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right) \right\| \in [(1 - \delta)d_F(u, v), (1 + \delta)d_F(u, v)].$$

To see this, observe that,

$$\begin{aligned} d_{\Gamma(F)}(u, v) &= \left\| \frac{\Gamma(F(u))}{\|\Gamma(F(u))\|} - \frac{\Gamma(F(v))}{\|\Gamma(F(v))\|} \right\| \\ &\geq \left\| \frac{\Gamma(F(u))}{\|F(u)\|} - \frac{\Gamma(F(v))}{\|F(v)\|} \right\| - \left\| \frac{\Gamma(F(u))}{\|\Gamma(F(u))\|} - \frac{\Gamma(F(u))}{\|F(u)\|} \right\| - \left\| \frac{\Gamma(F(v))}{\|\Gamma(F(v))\|} - \frac{\Gamma(F(v))}{\|F(v)\|} \right\| \\ &\geq \left\| \Gamma \left( \frac{F(u)}{\|F(u)\|} - \frac{F(v)}{\|F(v)\|} \right) \right\| - 2\delta \\ &\geq (1 - \delta)d_F(u, v) - 2\delta, \end{aligned}$$

where we have used the fact that  $\Gamma$  is a linear operator. The other direction can be proved similarly.

Therefore, by (21), and a union bound, for any  $u, v \in V$ ,  $\mathbb{P}[I_{u,v}] \leq 3\delta k^{-3}/128$ . Let,

$$\mathcal{E}_I := \sum_{u, v \in V} w(u)w(v) \|F(u)\|^2 \|F(v)\|^2 I_{u,v}.$$

By linearity of expectation, and Markov's inequality, we conclude that

$$\mathbb{P} \left[ \mathcal{E}_I \geq \frac{\delta}{4k^3} \left( \sum_{v \in V} w(v) \|F(v)\|^2 \right)^2 \right] \leq \frac{1}{8}. \quad (27)$$

Now suppose there exists a subset  $S \subseteq U$  with  $\text{diam}(S, d_{\Gamma(F)}) \leq \Delta/4$  and

$$\sum_{v \in S} w(v) \|\Gamma(F(v))\|^2 \geq (1 + 6\delta)\eta \sum_{v \in V} w(v) \|\Gamma(F(v))\|^2.$$

Fix a vertex  $u \in S$ . Since for every  $v \in S \setminus B_{d_F}(u, \Delta/2)$ , we have  $d_F(u, v) \geq \Delta/2$ ,  $d_{\Gamma(F)}(u, v) \leq \Delta/4$ , and recalling that  $\delta = \Delta/16$ , it must be that  $I_{u,v} = 1$ . On the other hand, we have

$$\begin{aligned}
\sum_{v \in S \setminus B_{d_F}(u, \Delta/2)} w(v) \|F(v)\|^2 &\geq \sum_{v \in S} w(v) \|F(v)\|^2 - \sum_{v \in B_{d_F}(u, \Delta/2)} w(v) \|F(v)\|^2 \\
&\geq (1 - \delta) \sum_{v \in S} w(v) \|\Gamma(F(v))\|^2 - \eta \sum_{v \in V} w(v) \|F(v)\|^2 \\
&\geq (1 - \delta)(1 + 6\delta)\eta \sum_{v \in V} w(v) \|\Gamma(F(v))\|^2 - \eta \sum_{v \in V} w(v) \|F(v)\|^2 \\
&\stackrel{(26)}{\geq} [(1 - 2\delta)(1 - \delta)(1 + 6\delta) - 1] \eta \sum_{v \in V} w(v) \|F(v)\|^2 \\
&\geq \delta \eta \sum_{v \in V} w(v) \|F(v)\|^2,
\end{aligned}$$

where we have used the fact that  $S \subseteq U$  and also  $\text{diam}(B_{d_F}(u, \Delta/2)) \leq \Delta$  and the fact that  $F$  is  $(\Delta, \eta)$ -spreading. In the final line, we have used  $\delta \leq 1/16$ .

Thus under our assumption on the existence of  $S$  and again using  $S \subseteq U$ , we have

$$\begin{aligned}
\mathcal{E}_I &\geq \sum_{u \in S} w(u) \|F(u)\|^2 \sum_{v \in S \setminus B_{d_F}(u, \Delta/2)} w(v) \|F(v)\|^2 \\
&\geq \delta \eta \left( \sum_{v \in V} w(v) \|F(v)\|^2 \right) \sum_{u \in S} w(u) \|F(u)\|^2 \\
&\geq \delta \eta \left( \sum_{v \in V} w(v) \|F(v)\|^2 \right) (1 - \delta) \sum_{u \in S} w(u) \|\Gamma(F(u))\|^2 \\
&\geq \delta(1 - \delta)\eta \left( \sum_{v \in V} w(v) \|F(v)\|^2 \right) (1 + 6\delta)\eta \left( \sum_{v \in V} w(v) \|\Gamma(F(v))\|^2 \right) \\
&\stackrel{(26)}{\geq} \delta(1 - \delta)(1 - 2\delta)(1 + 6\delta)\eta^2 \left( \sum_{v \in V} w(v) \|F(v)\|^2 \right)^2 \\
&\geq \frac{\delta}{4k^3} \left( \sum_{v \in V} w(v) \|F(v)\|^2 \right)^2,
\end{aligned}$$

where the last inequality follows from  $\eta \geq 1/k$  and  $\delta \leq 1/16$ . Combining this with (27) yields the claim.  $\square$

The preceding claim guarantees a spreading property for subsets  $S \subseteq U$ . Finally, we need to handle points outside  $U$ .

**Claim 4.5.** *With probability at least 15/16, we have*

$$\sum_{v \notin U} w(v) \|\Gamma(F(v))\|^2 \leq \delta k^{-3} \sum_{v \in V} w(v) \|F(v)\|^2.$$

*Proof.* Let  $\mathcal{D}_u$  be the event that  $u \notin U$ , and let  $H_u := \|\Gamma(F(u))\|^2 \mathbf{1}_{\mathcal{D}_u}$ . Then,

$$\mathbb{E} \left[ \sum_{u \notin U} w(u) \|\Gamma(F(u))\|^2 \right] = \sum_{u \in V} w(u) \mathbb{E}[H_u]. \quad (28)$$

Now we can estimate,

$$\frac{\mathbb{E}[H_u]}{\|F(u)\|^2} \leq 2 \mathbb{P}(\mathcal{D}_u) + \mathbb{P} \left( \frac{\|\Gamma(F(u))\|^2}{\|F(u)\|^2} > 2 \right) \cdot \mathbb{E} \left[ \frac{\|\Gamma(F(u))\|^2}{\|F(u)\|^2} \mid \|\Gamma(F(u))\|^2 > 2 \|F(u)\|^2 \right]. \quad (29)$$

Using the inequality, valid for all non-negative  $X$ ,

$$\mathbb{P}(X > \lambda_0) \cdot \mathbb{E}[X \mid X > \lambda_0] \leq \int_{\lambda_0}^{\infty} \lambda \cdot \mathbb{P}(X > \lambda) d\lambda,$$

we can bound the latter term in (29) by,

$$\int_2^{\infty} \lambda \cdot \mathbb{P} \left( \|\Gamma(F(u))\|^2 > \lambda \|F(u)\|^2 \right) d\lambda \leq \int_2^{\infty} \lambda e^{-\lambda h/12} d\lambda = \left( \frac{24}{h} + \frac{144}{h^2} \right) e^{-h/6} \leq \frac{\delta}{128k^3},$$

where we have used (22) and the initial choice of  $h$  sufficiently large.

It follows from this, (29), and (24), that

$$\mathbb{E}[H_u] \leq \frac{3\delta}{128k^3} \|F(u)\|^2.$$

Therefore, by Markov's inequality,

$$\mathbb{P} \left[ \sum_{v \notin U} w(v) \|\Gamma(F(v))\|^2 > \delta k^{-3} \sum_{v \in V} w(v) \|F(v)\|^2 \right] \leq \frac{3}{128},$$

completing the proof.  $\square$

To conclude the proof of the lemma, we need to verify that (ii) holds with probability at least  $25/32$ . But observe that if (26) holds, then the conclusion of the preceding claim is,

$$\sum_{v \notin U} w(v) \|\Gamma(F(v))\|^2 \leq \delta k^{-3} \sum_{v \in V} w(v) \|F(v)\|^2 \leq 2\delta k^{-3} \sum_{v \in V} w(v) \|\Gamma(F(v))\|^2 \leq \delta \eta \sum_{v \in V} w(v) \|\Gamma(F(v))\|^2.$$

Combining this with Claim 4.4 shows that with probability at least  $25/32$ ,  $\Gamma \circ F$  is  $(\Delta/4, (1+7\delta)\eta)$ -spreading, completing the proof.  $\square$

As an application of the preceding lemma, observe that we can improve (16) in Theorem 3.7 to the following bound, which is sometimes stronger, using the essentially same proof, but first obtaining a spreading representation  $F : V \rightarrow \mathbb{R}^{O(\delta^{-2} \log k)}$  using Lemma 4.3.

**Theorem 4.6.** *For any weighted graph  $G = (V, E, w)$  and  $\delta > 0$  the following holds. For every  $k \in \mathbb{N}$ , there exist  $r \geq \lceil (1-\delta)k \rceil$  disjointly supported functions  $\psi_1, \psi_2, \dots, \psi_r : V \rightarrow \mathbb{R}$  such that*

$$\mathcal{R}_G(\psi_i) \lesssim \delta^{-7} \log^2(k+1) \lambda_k. \quad (30)$$

where  $\lambda_k$  is the  $k$ th smallest eigenvalue of  $\mathcal{L}_G$ .

*Proof.* Let  $f_1, f_2, \dots, f_k : V \rightarrow \mathbb{R}$  be an  $\ell^2(V, w)$ -orthonormal system of eigenfunctions corresponding to the first  $k$  eigenvalues of  $\mathcal{L}_G$ , and define  $F : V \rightarrow \mathbb{R}^k$  by  $F(v) = (f_1(v), f_2(v), \dots, f_k(v))$ .

We may clearly assume that  $\delta \geq \frac{1}{2k}$ . Choose  $\Delta \asymp \delta$  so that  $(1 - 16\Delta^2)^{-1}(1 + 4\Delta) \leq 1 + \frac{\delta}{48}$ . In this case, for some choice of

$$h \lesssim \frac{1 + \log(k) + \log\left(\frac{1}{\Delta}\right)}{\Delta^2} \lesssim \frac{O(\log k)}{\delta^2},$$

with probability at least  $1/2$ ,  $\Gamma_{k,h}$  satisfies the conclusions of Lemma 4.3. Assume that  $\Gamma : \mathbb{R}^k \rightarrow \mathbb{R}^h$  is some map satisfying these conclusions.

Then combining (ii) from Lemma 4.3 with Lemma 3.2, we see that  $\Gamma \circ F : V \rightarrow \mathbb{R}^h$  is  $(\Delta, \frac{1}{k} + \frac{\delta}{48k})$ -spreading. Now we finish as in the proof of Theorem 3.7, using the fact that  $h = O(\delta^{-2} \log k)$ .  $\square$

## 4.2 A multi-way Cheeger inequality

Note that Theorem 4.6 combined with Lemma 2.1 is still not strong enough to prove Theorem 4.1. To do that, we need to combine Lemma 4.3 with a strong Cheeger inequality for Lipschitz partitions.

Let  $G = (V, E, w)$  be a weighted graph, and  $F : V \rightarrow \mathbb{R}^h$ . Set  $M = \max\{\|F(v)\|^2 : v \in V\}$ . Let  $\tau \in (0, M)$  be chosen uniformly at random, and for any subset  $S \subseteq V$ , define

$$\hat{S} = \{v \in S : \|F(v)\|^2 \geq \tau\}.$$

**Lemma 4.7.** *For every  $\Delta > 0$ , there exists a partition  $V = S_1 \cup S_2 \cup \dots \cup S_m$  such that for every  $i \in [m]$ ,  $\text{diam}(S_i, d_F) \leq \Delta$ , and*

$$\frac{\mathbb{E} \left[ w(E(\hat{S}_1, \overline{\hat{S}_1})) + w(E(\hat{S}_2, \overline{\hat{S}_2})) + \dots + w(E(\hat{S}_m, \overline{\hat{S}_m})) \right]}{\mathbb{E} \left[ w(\hat{S}_1) + \dots + w(\hat{S}_m) \right]} \lesssim \frac{\sqrt{h}}{\Delta} \sqrt{\mathcal{R}_G(F)}. \quad (31)$$

*Proof.* Since the statement of the lemma is homogeneous in  $F$ , we may assume that  $M = 1$ . By Theorem 2.3, there exists an  $\Delta$ -bounded random partition  $\mathcal{P}$  satisfying, for every  $u, v \in V$ ,

$$\mathbb{P}(\mathcal{P}(u) \neq \mathcal{P}(v)) \lesssim \frac{\sqrt{h}}{\Delta} \cdot d_F(u, v). \quad (32)$$

Let  $\mathcal{P} = S_1 \cup S_2 \cup \dots \cup S_m$ , where we recall that  $m$  is a random number.

First, observe that,  $\mathbb{E}[w(\hat{S}_i)] = \sum_{v \in S_i} w(v) \|F(v)\|^2$ , thus,

$$\mathbb{E} \left[ w(\hat{S}_1) + \dots + w(\hat{S}_m) \right] = \sum_{v \in V} w(v) \|F(v)\|^2. \quad (33)$$

Next, if  $\{u, v\} \in E$  with  $\|F(u)\|^2 \leq \|F(v)\|^2$ , then we have

$$\begin{aligned}
& \mathbb{P} \left[ \{u, v\} \in E(\hat{S}_1, \overline{\hat{S}_1}) \cup \dots \cup E(\hat{S}_m, \overline{\hat{S}_m}) \right] \\
& \leq \mathbb{P} [\mathcal{P}(u) \neq \mathcal{P}(v)] \cdot \mathbb{P} [\|F(u)\|^2 \geq \tau \text{ or } \|F(v)\|^2 \geq \tau \mid \mathcal{P}(u) \neq \mathcal{P}(v)] \\
& \quad + \mathbb{P} [\tau \in [\|F(u)\|^2, \|F(v)\|^2] \mid \mathcal{P}(u) = \mathcal{P}(v)] \\
& \lesssim \frac{\sqrt{h}}{\Delta} \cdot d_F(u, v) (\|F(u)\|^2 + \|F(v)\|^2) + \|F(v)\|^2 - \|F(u)\|^2 \\
& \leq (\|F(u)\| + \|F(v)\|) \left( \frac{\sqrt{h}}{\Delta} \cdot d_F(u, v) (\|F(u)\| + \|F(v)\|) + \|F(v)\| - \|F(u)\| \right) \\
& \leq \frac{5\sqrt{h}}{\Delta} (\|F(u)\| + \|F(v)\|) \|F(u) - F(v)\|,
\end{aligned}$$

where in the final line we have used Lemma 3.1.

Thus, we can use Cauchy-Schwarz to write,

$$\begin{aligned}
\mathbb{E} \left[ w(E(\hat{S}_1, \overline{\hat{S}_1})) + \dots + w(E(\hat{S}_m, \overline{\hat{S}_m})) \right] & \lesssim \frac{\sqrt{h}}{\Delta} \sum_{u \sim v} w(u, v) (\|F(u)\| + \|F(v)\|) \|F(u) - F(v)\| \\
& \leq \frac{\sqrt{h}}{\Delta} \sqrt{\sum_{u \sim v} w(u, v) (\|F(u)\| + \|F(v)\|)^2} \\
& \quad \cdot \sqrt{\sum_{u \sim v} w(u, v) \|F(u) - F(v)\|^2} \\
& \leq \frac{\sqrt{h}}{\Delta} \sqrt{2 \sum_{v \in V} w(v) \|F(v)\|^2} \sqrt{\sum_{u \sim v} w(u, v) \|F(u) - F(v)\|^2}.
\end{aligned}$$

Combining this with (33) yields,

$$\frac{\mathbb{E}_{\mathcal{P}} \mathbb{E} \left[ w(E(\hat{S}_1, \overline{\hat{S}_1})) + \dots + w(E(\hat{S}_m, \overline{\hat{S}_m})) \right]}{\mathbb{E} \left[ w(\hat{S}_1) + \dots + w(\hat{S}_m) \right]} \lesssim \frac{\sqrt{h}}{\Delta} \sqrt{\frac{\sum_{u \sim v} w(u, v) \|F(u) - F(v)\|^2}{\sum_{v \in V} w(v) \|F(v)\|^2}},$$

where we use  $\mathbb{E}_{\mathcal{P}}$  to denote expectation over the random choice of  $\mathcal{P}$ . In particular, there must exist a single partition  $P$  satisfying the statement of the lemma.  $\square$

We can use the preceding theorem to find many non-expanding sets, assuming that  $F : V \rightarrow \mathbb{R}^h$  has sufficiently good spreading properties.

**Lemma 4.8.** *Let  $G = (V, E, w)$  be a weighted graph and let  $k \in \mathbb{N}$  and  $\delta \in (0, 1)$  be given. If the map  $F : V \rightarrow \mathbb{R}^h$  is  $(\Delta, \frac{1}{k} + \frac{\delta}{4k})$ -spreading, then there exist  $r \geq \lceil (1-\delta)k \rceil$  disjoint sets  $T_1^*, T_2^*, \dots, T_r^*$ , such that*

$$\phi_G(T_i^*) \lesssim \frac{\sqrt{h}}{\delta \Delta} \sqrt{\mathcal{R}_G(F)}.$$

*Proof.* Let  $V = S_1 \cup S_2 \cup \dots \cup S_m$  be the partition guaranteed by applying Lemma 4.7 to the mapping  $F : V \rightarrow \mathbb{R}^h$ . Set  $\mathcal{E} := \sum_{v \in V} w(v) \|F(v)\|^2$ . Since  $F$  is  $(\Delta, \frac{1}{k} + \frac{\delta}{4k})$ -spreading and each  $S_i$  satisfies  $\text{diam}(S_i, d_F) \leq \Delta$ , we can form  $r' \geq \lceil (1 - \delta/2)k \rceil$  sets  $T_1, T_2, \dots, T_{r'}$  by taking disjoint unions of the sets  $\{S_i\}$  so that for each  $i = 1, 2, \dots, r'$ , we have

$$\frac{\mathcal{E}}{2k} \leq \sum_{v \in T_i} w(v) \|F(v)\|^2 \leq \frac{\mathcal{E}}{k} \left(1 + \frac{\delta}{4}\right).$$

In particular,  $\mathbb{E}[w(\hat{T}_i)] = \sum_{v \in T_i} w(v) \|F(v)\|^2 \in [\frac{1}{2} \frac{\mathcal{E}}{k}, (1 + \frac{\delta}{4}) \frac{\mathcal{E}}{k}]$ .

Order the sets so that  $\mathbb{E}[w(E(\hat{T}_i, \overline{\hat{T}_i}))] \leq \mathbb{E}[w(E(\hat{T}_{i+1}, \overline{\hat{T}_{i+1}}))]$  for  $i = 1, 2, \dots, r' - 1$ , and let  $r = \lceil (1 - \delta)k \rceil$ . Then from (31), it must be that each  $i = 1, 2, \dots, r$  satisfies

$$\begin{aligned} \mathbb{E}[w(E(\hat{T}_i, \overline{\hat{T}_i}))] &\lesssim \frac{1}{\delta k} \mathbb{E} \left[ \sum_{j=1}^m w(E(\hat{S}_j, \overline{\hat{S}_j})) \right] \\ &\lesssim \frac{\sqrt{h}}{\delta k \cdot \Delta} \cdot \sqrt{\mathcal{R}_G(F)} \cdot \mathbb{E} \left[ \sum_{j=1}^m w(\hat{S}_j) \right] \lesssim \frac{\sqrt{h}}{\delta k \cdot \Delta} \cdot \sqrt{\mathcal{R}_G(F)} \cdot \mathcal{E}. \end{aligned}$$

But  $\mathbb{E}[w(\hat{T}_i)] \asymp \mathcal{E}/k$  for each  $i = 1, 2, \dots, r$ , showing that

$$\frac{\mathbb{E}[w(E(\hat{T}_i, \overline{\hat{T}_i}))]}{\mathbb{E}[w(\hat{T}_i)]} \lesssim \frac{\sqrt{h}}{\delta \Delta} \cdot \sqrt{\mathcal{R}_G(F)}.$$

□

We can already use this to improve (19) in Theorem 3.9.

**Theorem 4.9.** *For every  $\delta \in (0, 1)$  and any weighted graph  $G = (V, E, w)$ , there exist  $r \geq \lceil (1 - \delta)k \rceil$  disjoint, non-empty sets  $S_1, S_2, \dots, S_r \subseteq V$  such that,*

$$\phi_G(S_i) \lesssim \frac{\sqrt{k}}{\delta^{3/2}} \sqrt{\lambda_k}. \quad (34)$$

where  $\lambda_k$  is the  $k$ th smallest eigenvalue of  $\mathcal{L}_G$ .

*Proof.* Let  $\Delta \asymp \sqrt{\delta}$  be such that  $(1 - \Delta^2)^{-1} \leq 1 + \frac{\delta}{4}$ . If we take  $F : V \rightarrow \mathbb{R}^k$  to be the embedding coming from the first  $k$  eigenfunctions of  $\mathcal{L}_G$ , then Lemma 3.2 implies that  $F$  is  $(\Delta, \frac{1}{k} + \frac{\delta}{4k})$ -spreading. Now apply Lemma 4.8. □

Observe that setting  $\delta = \frac{1}{2k}$  in the preceding theorem yields Theorem 1.1.

And now we can complete the proof of Theorem 4.1.

*Proof of Theorem 4.1.* Let  $F(v) = (f_1(v), f_2(v), \dots, f_k(v))$ . Choose  $\Delta \asymp \delta$  so that  $(1 - 16\Delta^2)^{-1}(1 + 4\Delta) \leq 1 + \frac{\delta}{4}$ . In this case, for some choice of

$$h \lesssim \frac{1 + \log(k) + \log(\frac{1}{\Delta})}{\Delta^2} \lesssim \frac{O(\log k)}{\delta^2},$$

with probability at least  $1/2$ ,  $\Gamma_{k,h}$  satisfies the conclusions of Lemma 4.3. Assume that  $\Gamma : \mathbb{R}^k \rightarrow \mathbb{R}^h$  is some map satisfying these conclusions.

Then combining the conclusions of Lemma 4.3 with Lemma 3.2, we see that  $F^* := \Gamma$  is  $(\Delta, \frac{1}{k} + \frac{\delta}{4k})$ -spreading, takes values in  $\mathbb{R}^h$ , and satisfies  $\mathcal{R}_G(F^*) \leq 8 \cdot \mathcal{R}_G(F)$ . Now applying Lemma 4.8 yields the desired result.  $\square$

### 4.3 Noisy hypercubes

In the present section, we review examples for which Corollary 4.2 is tight. For  $k \in \mathbb{N}$  and  $\varepsilon \in (0, 1)$  let  $H_{k,\varepsilon} = (V, E)$  be the “noisy hypercube” graph, where  $V = \{0, 1\}^k$ , and for any  $x, y \in V$  there is an edge of weight  $w(x, y) = \varepsilon^{\|x-y\|_1}$ . We put  $n = |V| = 2^k$ .

**Theorem 4.10.** *For any  $1 \leq C < k$  and  $k \in \mathbb{N}$ , and  $S \subseteq V$  with  $|S| \leq Cn/k$ , we have*

$$\phi_{H_{k,\varepsilon}}(S) \gtrsim \sqrt{\lambda_k \log(k/C)},$$

where  $\varepsilon = \frac{\log(2)}{\log(k/C)}$ .

*Proof.* Let  $H = H_{k,\varepsilon}$ . First, the weighted degree of every vertex is

$$w(x) = \sum_{y \in V} \varepsilon^{\|x-y\|_1} = (1 + \varepsilon)^k.$$

Therefore, if we define  $F_i : V \rightarrow \mathbb{R}$  by  $F_i(x) = (-1)^{x_i}$ , then

$$\mathcal{R}_{H_{k,\varepsilon}}(F_i) = \frac{\sum_{\{x,y\}} w(x,y) |F_i(x) - F_i(y)|^2}{\sum_x w(x) F_i(x)^2} = \frac{2\varepsilon n(1 + \varepsilon)^{k-1}}{n(1 + \varepsilon)^k} \leq 2\varepsilon.$$

Thus  $\lambda_k(H) \leq 2\varepsilon$ . We will now show that for  $|S| \leq Cn/k$ , one has  $\phi_H(S) \geq \frac{1}{2}$ , completing the proof of the theorem.

To bound  $\phi_H(\cdot)$ , we need to recall some Fourier analysis. For  $f, g : \{0, 1\}^k \rightarrow \mathbb{R}$  define the inner product:

$$\langle f, g \rangle_{L^2(V)} := \frac{1}{n} \sum_{x \in \{0,1\}^k} f(x)g(x).$$

Given  $S \subseteq [k]$ , the Walsh function  $W_S : \{0, 1\}^k \rightarrow \mathbb{R}$  is defined by  $W_S(x) = (-1)^{\sum_{i \in S} x_i}$ . The Walsh functions form an orthonormal basis with respect to the above inner product. Therefore, any function  $f : \{0, 1\}^k \rightarrow \mathbb{R}$  has a unique representation as  $f = \sum_{S \subseteq [n]} \hat{f}(S) W_S$ , where  $\hat{f}(S) := \langle f, W_S \rangle_{L^2(V)}$ .

For  $\eta \in [0, 1]$ , the Bonami-Beckner operator  $T_\eta$  is defined as

$$T_\eta f := \sum_{S \subseteq [n]} \eta^{|S|} \hat{f}(S) W_S.$$

The Bonami-Beckner inequality [Bon70, Bec75] states that

$$\sum_{S \subseteq [n]} \eta^{|S|} \hat{f}(S)^2 = \|T_{\sqrt{\eta}} f\|_2^2 \leq \|f\|_{1+\eta}^2 = \left\{ \frac{1}{n} \sum_{x \in \{0,1\}^k} f(x)^{1+\eta} \right\}^{\frac{2}{1+\eta}}. \quad (35)$$



Let  $A$  be the normalized adjacency matrix of  $H$ , i.e.  $A_{xy} = \frac{\varepsilon^{|x \oplus y|}}{(1+\varepsilon)^k}$ . It follows from an elementary calculation that  $W_S$  is an eigenvector of  $A$  with eigenvalue  $(\frac{1-\varepsilon}{1+\varepsilon})^{|S|}$ , i.e.

$$AW_S = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{|S|} W_S.$$

For  $S \subseteq [n]$ , let  $\mathbf{1}_S$  be the indicator function of  $S$ . Therefore,

$$\langle \mathbf{1}_S, A\mathbf{1}_S \rangle_{L^2(V)} = \sum_{T \subseteq [n]} \hat{\mathbf{1}}_S(T)^2 \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{|T|} \leq \|\mathbf{1}_S\|_{\frac{2}{1+\varepsilon}}^2 = \left(\frac{|S|}{n}\right)^{1+\varepsilon},$$

where the one last inequality follows from (35).

Now, observe that for any  $S \subseteq V$ , we have

$$w(E(S, \bar{S})) = w(S) - w(E(S, S)) = w(S) - (1+\varepsilon)^k n \langle \mathbf{1}_S, A\mathbf{1}_S \rangle_{L^2(V)}$$

where we have written  $E(S, S)$  for edges with both endpoints in  $S$ .

Hence, for any subset  $S \subseteq V$  of size  $|S| \leq Cn/k$ , we have

$$\phi_H(S) = \frac{w(E(S, \bar{S}))}{w(S)} = \frac{|S| - n \langle \mathbf{1}_S, A\mathbf{1}_S \rangle_{L^2(V)}}{|S|} \geq 1 - \left(\frac{|S|}{n}\right)^\varepsilon \geq 1 - (k/C)^{-\varepsilon} \geq \frac{1}{2}$$

where the last inequality follows by the choice of  $\varepsilon = \log(2)/\log(k/C)$ .  $\square$

**Remark 4.1.** *The preceding theorem shows that even if we only want to find a set  $S$  of size  $n/\sqrt{k}$ , then for values of  $k \leq O(\log n)$ , we can still only achieve a bound of the form  $\phi_H(S) \lesssim \sqrt{\lambda_k \log k}$ . The state of affairs for  $k \gg \log n$  is a fascinating open question.*

## 5 Conclusion

In Section 1.2, we gave a generic outline of our spectral partitioning algorithm. We remark that our instantiations of this algorithm are simple to describe. As an example, suppose we are given a weighted graph  $G = (V, E, w)$  and want to find  $k$  disjoint sets, each of expansion  $O(\sqrt{\lambda_{2k} \log k})$  (recall Theorem 1.2). We specify a complete randomized algorithm.

One starts with the spectral embedding  $F : V \rightarrow \mathbb{R}^k$ , given by  $F(v) = (f_1(v), f_2(v), \dots, f_{2k}(v))$ , where  $f_1, f_2, \dots, f_{2k}$  is the  $\ell^2(V, w)$ -orthogonal system comprised of the first  $2k$  eigenfunctions of the normalized Laplacian. Then, for some  $h = O(\log k)$ , we perform random projection into  $\mathbb{R}^h$ . Let  $\Gamma_{2k,h} : \mathbb{R}^{2k} \rightarrow \mathbb{R}^h$  be the random linear map given by

$$\Gamma_{2k,h}(x) = \frac{1}{\sqrt{h}} (\langle g_1, x \rangle, \dots, \langle g_h, x \rangle),$$

where  $\{g_1, \dots, g_h\}$  are i.i.d. standard Gaussians. We now have an embedding  $F^* := \Gamma_{2k,h} \circ F : V \rightarrow \mathbb{R}^h$ .

Next, for some  $R = \Theta(1)$ , we perform the random space partitioning algorithm from [CCGG98]. Let  $\mathcal{B}$  denotes the closed Euclidean unit ball in  $\mathbb{R}^h$ . Consider  $V \subseteq \mathcal{B}$  by identifying each vertex

with its image under the map  $v \mapsto F^*(v)/\|F^*(v)\|$ . If  $\{x_1, x_2, \dots\}$  is an i.i.d. sequence of points in  $\mathcal{B}$  (chosen according to the Lebesgue measure), then we form a partition of  $V$  into the sets

$$V = \bigcup_{i=1}^{\infty} \left[ V \cap B(x_i, R) \setminus (B(x_1, R) \cup \dots \cup B(x_{i-1}, R)) \right]$$

Here,  $B(x, R)$  represents the closed Euclidean ball of radius  $R$  about  $x$ , and it is easy to see that this induces a partition of  $V$  in a finite number of steps with probability one. Let  $V = S_1 \cup S_2 \cup \dots \cup S_m$  be this partition.

Finally, for a subset  $S \subseteq V$ , let  $\mathcal{E}(S) = \sum_{v \in S} w(v) \|F^*(v)\|^2$ . We sort the partition  $\{S_1, S_2, \dots, S_m\}$  in decreasing order according to  $\mathcal{E}(S_i)$ . Let  $k' = \lceil \frac{3}{2}k \rceil$ . Then for each  $i = k' + 1, k' + 2, \dots, m$ , we iteratively set  $S_\ell := S_\ell \cup S_i$  where

$$\ell = \operatorname{argmin}\{\mathcal{E}(S_j) : j \leq k'\}.$$

(Intuitively, we form  $k'$  sets from our total of  $m \geq k'$  sets by balancing the  $\mathcal{E}(\cdot)$ -value among them.) At the end, we are left with a partition  $V = S_1 \cup S_2 \cup \dots \cup S_{k'}$  of  $V$  into  $k' \geq 3k/2$  sets.

To complete the algorithm, for each  $i = 1, 2, \dots, k'$ , we choose a value  $\tau$  such that

$$\hat{S}_i = \{v \in S_i : \|F^*(v)\|^2 \geq \tau\}$$

has the least expansion. We then output  $k$  of the sets  $\hat{S}_1, \hat{S}_2, \dots, \hat{S}_{k'}$  that have the smallest expansion.

The preceding algorithm suggests some natural questions. First, does dimension reduction help to improve the quality of clusterings in practice? For instance, if one runs the  $k$ -means algorithm (as in [NJW02]) on the randomly projected points, does it yield better results? Another interesting question is whether, at least in certain circumstances, the quality of the  $k$ -means clustering can be rigorously analyzed when used in place of our random geometric partitioning.

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